Assignment 1

4DM10, Multibody and Nonlinear Dynamics - Q2 (2025)

Group 17

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1 | Problem 1: Modeling of the Floating Telescopic Crane

1.1 | Degrees of Freedom

To determine the number of degrees of freedom (DOF), we analyze the system using Grübler's formula for planar mechanisms:

$$\text{DOF} = 3N_{\text{bodies}} - N_{\text{constraints}} \tag{1.1}$$

The system consists of $N_{\text{bodies}} = 3$ rigid bodies in planar motion, giving $3 \times 3 = 9$ potential DOF before constraints are applied.
Constraint Analysis:

- **Body 1**: Since the center of mass CM_1 must remain on the vertical axis $\vec{e}^{\,0}_2$, meaning there is no horizontal translation, one translation is constrained.
- **Revolute Joint (Body 1–Body 2) at point A**: This joint requires that the position of point A computed from Body 1 must equal the position computed from Body 2. This gives 2 independent position constraints (x and y component must match). Therefore, another 2 constraints appear.
- **Prismatic Joint (Body 2–Body 3)**: The prismatic joint presents 2 more constraints, blocking any relative rotation($\theta_3 = \theta_2$) and any relative translation perpendicular to the sliding axis $\vec{e}^{\,3}_1$.

This results in the total number of constraints: $N_{\text{constraints}} = 1 + 2 + 2 = 5$. Therefore, total number of DOF = 9 - 5 = 4
The generalized coordinate vecor:

$$\mathbf{q} = [x_1, y_1, \theta_1, x_2, y_2, \theta_2, x_3, y_3, \theta_3]^T \tag{1.2}$$

contains 9 coordinates. Since the system has only 4 DOF, the coordinates in $\mathbf{q}$ are dependent. These 9 coordinates are related through the 5 holonomic constraint equations that will be derived in Problem 1.2.

1.2 | Constraint Equations

We now know there are five constraints, so let our constraint vecor be:

$$\mathbf{h}(\mathbf{q}) = \begin{bmatrix} h_1(\mathbf{q}) & h_2(\mathbf{q}) & h_3(\mathbf{q}) & h_4(\mathbf{q}) & h_5(\mathbf{q}) \end{bmatrix}^\top = \mathbf{0}. \tag{1.3}$$

Constraint 1 - Boat horizontal translation:
The kinematic constraint equations $\mathbf{h}(\mathbf{q}) = \mathbf{0}$ are derived as follows:

$$h_1(\mathbf{q}) = x_1 = 0 \tag{1.4}$$

This ensures that CM_1 remains aligned with the vertical axis $\vec{e}^{\,0}_2$.
Constraints 2 & 3 - Revolute joint at point A:
The revolute joint at A connects Body 1 and Body 2. For a revolute joint, the position of the joint point must be the same when computed from either body.
Position of A from Body 1 (using the body-fixed frame $\vec{e}^{\,1}$):

$$\mathbf{r}_A^{(1)} = \mathbf{r}_{CM_1} + \mathbf{R}_1 \begin{bmatrix} 0 \\ h_2 \end{bmatrix}, \text{where the scalar } h_2 \text{ should not be confused with second constraint } h_2(\mathbf{q}) \tag{1.5}$$

where $\mathbf{R}_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$ is the rotation matrix for Body 1.
Expanding:

$$\mathbf{r}_A^{(1)} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} -h_2 \sin\theta_1 \\ h_2 \cos\theta_1 \end{bmatrix} = \begin{bmatrix} x_1 - h_2 \sin\theta_1 \\ y_1 + h_2 \cos\theta_1 \end{bmatrix} \tag{1.6}$$

Position of A from Body 2 (the joint is at distance ℓ_1 from CM_2 in the negative $\vec{e}^{\,2}_1$ direction):

$$\mathbf{r}_A^{(2)} = \mathbf{r}_{CM_2} + \mathbf{R}_2 \begin{bmatrix} -\ell_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} -\ell_1 \cos\theta_2 \\ -\ell_1 \sin\theta_2 \end{bmatrix} \tag{1.7}$$

Setting $\mathbf{r}_A^{(1)} = \mathbf{r}_A^{(2)}$:

$$h_2(\mathbf{q}) = x_1 - h_2 \sin\theta_1 - x_2 + \ell_1 \cos\theta_2 = 0 \tag{1.8}$$
$$h_3(\mathbf{q}) = y_1 + h_2 \cos\theta_1 - y_2 + \ell_1 \sin\theta_2 = 0 \tag{1.9}$$

Constraint 4 - Prismatic joint:

$$h_4(\mathbf{q}) = \theta_3 - \theta_2 = 0 \tag{1.10}$$

Body 3 remains aligned with the prismatic guide fixed to body 2 (no relative rotation).
Constraint 5 - Prismatic joint collinearity:
The center of mass CM_3 must lie on the axis of Body 2 (along $\vec{e}_1^2$). The vecor from CM_2 to CM_3 must be parallel to $\vec{e}_1^2 = [\cos\theta_2, \sin\theta_2]^T$. This means the component perpendicular to this axis must be zero:

$$h_5(\mathbf{q}) = -\sin\theta_2(x_3 - x_2) + \cos\theta_2(y_3 - y_2) = 0 \tag{1.11}$$

To conclude, the five constraints above are holonomic, since they are expressed at configuration level as $h_i(\mathbf{q}) = 0$ and depend only on the generalized coordinates (not on velocities). Moreover, they are scleronomic, since they do not depend explicitly on time.

1.3 | Kinetic Energy

For each planar rigid body $i \in \{1, 2, 3\}$ the kinetic energy is

$$T_i = \frac{1}{2} m_i \dot{\mathbf{r}}_{CM_i}^\top \dot{\mathbf{r}}_{CM_i} + \frac{1}{2} J_i \dot{\theta}_i^2, \qquad \dot{\mathbf{r}}_{CM_i} = \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix}. \tag{1.12}$$

Since it has no rotational velocity, the point load m_L at point P contributes only translational kinetic energy

$$\begin{aligned} T_L &= \frac{1}{2} m_L \, \dot{\mathbf{r}}_P^\top \dot{\mathbf{r}}_P, \\ \mathbf{r}_P &= \mathbf{r}_{CM_3} + \mathbf{R}_3 \, \mathbf{r}_{P/CM_3}^{(3)}, \qquad \text{with } \mathbf{r}_{P/CM_3}^{(3)} = \begin{bmatrix} \ell_4 \\ 0 \end{bmatrix} \end{aligned} \tag{1.13}$$

Hence, the total kinetic energy equals

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{i=1}^{3} \left[\frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2) + \frac{1}{2} J_i \dot{\theta}_i^2 \right] + \frac{1}{2} m_L \, \dot{\mathbf{r}}_P^\top \dot{\mathbf{r}}_P. \tag{1.14}$$

Body 1-3:
The kinetic energy of each of the three bodies is:

$$T_1 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} J_1 \dot{\theta}_1^2 \tag{1.15}$$

$$T_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} J_2 \dot{\theta}_2^2 \tag{1.16}$$

$$T_3 = \frac{1}{2} m_3 (\dot{x}_3^2 + \dot{y}_3^2) + \frac{1}{2} J_3 \dot{\theta}_3^2 \tag{1.17}$$

Load at point P:
The load is a point mass m_L located at point P, which is at distance ℓ_4 from CM_3 along $\vec{e}_1^3$:

$$\mathbf{r}_P = \mathbf{r}_{CM_3} + \ell_4 \vec{e}_1^3 = \begin{bmatrix} x_3 + \ell_4 \cos\theta_3 \\ y_3 + \ell_4 \sin\theta_3 \end{bmatrix} \tag{1.18}$$

Taking the time derivative to compute the velocity:

$$\mathbf{v}_P = \dot{\mathbf{r}}_P = \begin{bmatrix} \dot{x}_3 - \ell_4 \sin\theta_3 \dot{\theta}_3 \\ \dot{y}_3 + \ell_4 \cos\theta_3 \dot{\theta}_3 \end{bmatrix} \tag{1.19}$$

The kinetic energy of the load is:

$$T_L = \frac{1}{2} m_L \|\mathbf{v}_P\|^2 = \frac{1}{2} m_L \left[(\dot{x}_3 - \ell_4 \sin\theta_3 \dot{\theta}_3)^2 + (\dot{y}_3 + \ell_4 \cos\theta_3 \dot{\theta}_3)^2 \right] \tag{1.20}$$

The total kinetic energy can be written in matrix form:

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} \tag{1.21}$$

where the mass matrix $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{9\times 9}$ has the following non-zero elements:

$$M_{1,1} = m_1, \quad M_{2,2} = m_1, \quad M_{3,3} = J_1 \tag{1.22}$$
$$M_{4,4} = m_2, \quad M_{5,5} = m_2, \quad M_{6,6} = J_2 \tag{1.23}$$
$$M_{7,7} = m_3 + m_L, \quad M_{8,8} = m_3 + m_L \tag{1.24}$$
$$M_{9,9} = J_3 + m_L \ell_4^2 \tag{1.25}$$
$$M_{7,9} = M_{9,7} = -m_L \ell_4 \sin\theta_3 \tag{1.26}$$
$$M_{8,9} = M_{9,8} = m_L \ell_4 \cos\theta_3 \tag{1.27}$$

Using the holonomic constraints h_1–h_5, the motion can be described with the reduced coordinate vecor

$$\mathbf{q}_{\text{red}} = \begin{bmatrix} y_1 & \theta_1 & \theta_2 & s \end{bmatrix}^\top, \tag{1.28}$$

where s is the prismatic (telescopic) joint coordinate of body 3 relative to body 2 along $\vec{e}_1^{\,2}$. In particular, we parameterize the relative position as

$$\mathbf{r}_{CM_3} - \mathbf{r}_{CM_2} \triangleq s\,\vec{e}_1^{\,2}, \qquad \vec{e}_1^{\,2} = \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix}, \tag{1.29}$$

which satisfies the collinearity constraint $h_5(\mathbf{q}) = 0$ by construction. Kinematics after substitution of constraints
From $h_1(\mathbf{q}) = x_1 = 0$,

$$\mathbf{r}_{CM_1} = \begin{bmatrix} 0 \\ y_1 \end{bmatrix}, \qquad \dot{\mathbf{r}}_{CM_1} = \begin{bmatrix} 0 \\ \dot{y}_1 \end{bmatrix}. \tag{1.30}$$

From the revolute joint constraints $h_2(\mathbf{q}), h_3(\mathbf{q})$ at point A (i.e. $\mathbf{r}_A^{(1)} = \mathbf{r}_A^{(2)}$), the center of mass of body 2 becomes

$$\mathbf{r}_{CM_2} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -h_2 \sin\theta_1 + \ell_1 \cos\theta_2 \\ y_1 + h_2 \cos\theta_1 + \ell_1 \sin\theta_2 \end{bmatrix}. \tag{1.31}$$

Differentiating yields

$$\dot{x}_2 = -h_2 \cos\theta_1\, \dot{\theta}_1 - \ell_1 \sin\theta_2\, \dot{\theta}_2, \tag{1.32}$$
$$\dot{y}_2 = \dot{y}_1 - h_2 \sin\theta_1\, \dot{\theta}_1 + \ell_1 \cos\theta_2\, \dot{\theta}_2. \tag{1.33}$$

From the prismatic constraints $h_4(\mathbf{q}) : \theta_3 = \theta_2$ and $h_5(\mathbf{q})$ (collinearity along $\vec{e}_1^{\,2}$), we may write

$$\mathbf{r}_{CM_3} = \mathbf{r}_{CM_2} + s\,\vec{e}_1^{\,2} = \begin{bmatrix} x_2 + s\cos\theta_2 \\ y_2 + s\sin\theta_2 \end{bmatrix}, \qquad \dot{\theta}_3 = \dot{\theta}_2. \tag{1.34}$$

Hence,

$$\dot{x}_3 = \dot{x}_2 + \dot{s}\cos\theta_2 - s\sin\theta_2\, \dot{\theta}_2, \tag{1.35}$$
$$\dot{y}_3 = \dot{y}_2 + \dot{s}\sin\theta_2 + s\cos\theta_2\, \dot{\theta}_2. \tag{1.36}$$

The load point P is rigidly attached to body 3 at distance ℓ_4 along $+\vec{e}_1^{\,3}$. With h_4 we have $\vec{e}_1^{\,3} = \vec{e}_1^{\,2}$, and therefore

$$\mathbf{r}_P = \mathbf{r}_{CM_3} + \ell_4 \vec{e}_1^{\,3} = \mathbf{r}_{CM_2} + (s + \ell_4)\vec{e}_1^{\,2} = \begin{bmatrix} x_2 + (s+\ell_4)\cos\theta_2 \\ y_2 + (s+\ell_4)\sin\theta_2 \end{bmatrix}. \tag{1.37}$$

Differentiating gives

$$\dot{x}_P = \dot{x}_2 + \dot{s}\cos\theta_2 - (s+\ell_4)\sin\theta_2\, \dot{\theta}_2, \tag{1.38}$$
$$\dot{y}_P = \dot{y}_2 + \dot{s}\sin\theta_2 + (s+\ell_4)\cos\theta_2\, \dot{\theta}_2. \tag{1.39}$$

The total kinetic energy with constraints substituted is then

$$T(\mathbf{q}_{\text{red}}, \dot{\mathbf{q}}_{\text{red}}) = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) + \frac{1}{2}J_2\dot{\theta}_2^2 + \frac{1}{2}m_3\left(\dot{x}_3^2 + \dot{y}_3^2\right) + \frac{1}{2}J_3\dot{\theta}_2^2 + \frac{1}{2}m_L\left(\dot{x}_P^2 + \dot{y}_P^2\right), \tag{1.40}$$

where $\dot{x}_2, \dot{y}_2, \dot{x}_3, \dot{y}_3, \dot{x}_P, \dot{y}_P$ are given by the expressions above.

1.4 | Potential Energy

The potential energy consists of gravitational and elastic components:

$$V = V_{\text{grav}} + V_{\text{springs}} \tag{1.41}$$

Gravitational Potential Energy:
With $\vec{g} = -g\vec{e}_2^{\,0}$, the gravitational potential energy (taking sea level as reference) is:

$$V_{\text{grav}} = m_1 g y_1 + m_2 g y_2 + m_3 g y_3 + m_L g y_P \tag{1.42}$$

where $y_P = y_3 + \ell_4 \sin\theta_3$ is the vertical position of the load.

$$V_{\text{grav}} = m_1 g y_1 + m_2 g y_2 + (m_3 + m_L) g y_3 + m_L g \ell_4 \sin\theta_3 \tag{1.43}$$

Buoyancy Springs at H_1 and H_2:
The positions of H_1 and H_2 relative to CM_1 in the body-fixed frame are $[-b, -h_1]^T$ and $[b, -h_1]^T$ respectively.
The vertical positions of H_1 and H_2 in the global frame are:

$$y_{H_1} = y_1 + \left(\mathbf{R}_1 \begin{bmatrix} -b \\ -h_1 \end{bmatrix}\right)_2 = y_1 - b\sin\theta_1 - h_1\cos\theta_1 \tag{1.44}$$

$$y_{H_2} = y_1 + \left(\mathbf{R}_1 \begin{bmatrix} b \\ -h_1 \end{bmatrix}\right)_2 = y_1 + b\sin\theta_1 - h_1\cos\theta_1 \tag{1.45}$$

The spring potential energy (with zero unstretched length, measured from sea level $y = 0$):

$$V_{k_w} = \frac{1}{2}k_w y_{H_1}^2 + \frac{1}{2}k_w y_{H_2}^2 \tag{1.46}$$

Expanding:

$$V_{k_w} = \frac{1}{2}k_w(y_1 - b\sin\theta_1 - h_1\cos\theta_1)^2 + \frac{1}{2}k_w(y_1 + b\sin\theta_1 - h_1\cos\theta_1)^2 \tag{1.47}$$

Cable Spring k_1 between point B and CM_2:
Point B is located on Body 1 at height $(h_2 + \ell_1)$ above CM_1 along the body-fixed $\vec{e}_2^{\,1}$ axis:

$$\mathbf{r}_B = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \mathbf{R}_1 \begin{bmatrix} 0 \\ h_2 + \ell_1 \end{bmatrix} = \begin{bmatrix} x_1 - (h_2 + \ell_1)\sin\theta_1 \\ y_1 + (h_2 + \ell_1)\cos\theta_1 \end{bmatrix} \tag{1.48}$$

Since $x_1 = 0$:

$$\mathbf{r}_B = \begin{bmatrix} -(h_2 + \ell_1)\sin\theta_1 \\ y_1 + (h_2 + \ell_1)\cos\theta_1 \end{bmatrix} \tag{1.49}$$

The cable length is:

$$L_{\text{cable}} = \|\mathbf{r}_{CM_2} - \mathbf{r}_B\| = \sqrt{(x_2 + (h_2 + \ell_1)\sin\theta_1)^2 + (y_2 - y_1 - (h_2 + \ell_1)\cos\theta_1)^2} \tag{1.50}$$

Spring potential energy (zero unstretched length):

$$V_{k_1} = \frac{1}{2}k_1 L_{\text{cable}}^2 = \frac{1}{2}k_1\|\mathbf{r}_{CM_2} - \mathbf{r}_B\|^2 \tag{1.51}$$

Spring k_2 between CM_2 and CM_3:
Since the spring has zero unloaded length and at zero elongation the center of masses have an offset of$(\ell_2 + \ell_4)$, the spring potential energy becomes:

$$V_{k_2} = \frac{1}{2}k_2\left(s - \ell_2 - \ell_4\right)^2 = \frac{1}{2}k_2\left(\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} - \ell_2 - \ell_4\right)^2. \tag{1.52}$$

Defining the spring deformations as

$$\delta_{w_1} = y_{H_1}, \qquad \delta_{w_2} = y_{H_2}, \qquad \delta_1 = L_{\text{cable}}, \qquad \delta_2 = s - (\ell_2 + \ell_4), \tag{1.53}$$

and using $y_P = y_3 + \ell_4 \sin\theta_3$, the total potential energy can be written as:

$$V(\mathbf{q}) = m_1 g y_1 + m_2 g y_2 + m_3 g y_3 + m_L g y_P + \frac{1}{2} k_w \left(\delta_{w_1}^2 + \delta_{w_2}^2\right) + \frac{1}{2} k_1 \delta_1^2 + \frac{1}{2} k_2 \delta_2^2. \tag{1.54}$$

1.5 | Non-Conservative Generalized Forces

The non-conservative generalized forces are derived using virtual work: $\delta W = \mathbf{Q}_{\text{nc}}^T \delta \mathbf{q}$.
1. Wave Forces F_{w_1} and F_{w_2}:
These forces act vertically (along $\vec{e}_2^{\,0}$) at points H_1 and H_2.
The virtual displacement of H_1 is:

$$\delta y_{H_1} = \delta y_1 + (-b\cos\theta_1 + h_1 \sin\theta_1)\delta\theta_1 \tag{1.55}$$

Similarly for H_2:

$$\delta y_{H_2} = \delta y_1 + (b\cos\theta_1 + h_1 \sin\theta_1)\delta\theta_1 \tag{1.56}$$

Virtual work:

$$\delta W_{\text{wave}} = F_{w_1}\delta y_{H_1} + F_{w_2}\delta y_{H_2} \tag{1.57}$$

Contribution to generalized forces:

$$Q_{y_1}^{\text{wave}} = F_{w_1} + F_{w_2} \tag{1.58}$$
$$Q_{\theta_1}^{\text{wave}} = F_{w_1}(-b\cos\theta_1 + h_1\sin\theta_1) + F_{w_2}(b\cos\theta_1 + h_1\sin\theta_1) \tag{1.59}$$

2. Buoyancy Dampers at H_1 and H_2:
Vertical velocities of H_1 and H_2:

$$\dot{y}_{H_1} = \dot{y}_1 + (-b\cos\theta_1 + h_1\sin\theta_1)\dot{\theta}_1 \tag{1.60}$$
$$\dot{y}_{H_2} = \dot{y}_1 + (b\cos\theta_1 + h_1\sin\theta_1)\dot{\theta}_1 \tag{1.61}$$

Damping forces (opposing vertical velocity):

$$F_{d,H_1} = -d_w \dot{y}_{H_1}, \quad F_{d,H_2} = -d_w \dot{y}_{H_2} \tag{1.62}$$

Contributions to generalized forces:

$$Q_{y_1}^{\text{damp}} = -d_w(\dot{y}_{H_1} + \dot{y}_{H_2}) = -2d_w\dot{y}_1 - 2d_w h_1 \sin\theta_1 \dot{\theta}_1 \tag{1.63}$$
$$Q_{\theta_1}^{\text{damp}} = -d_w\dot{y}_{H_1}(-b\cos\theta_1 + h_1\sin\theta_1) - d_w\dot{y}_{H_2}(b\cos\theta_1 + h_1\sin\theta_1) \tag{1.64}$$

3. Motor Torque M:
The motor applies torque M on Body 2 relative to Body 1. By Newton's third law:

$$Q_{\theta_1}^{M} = -M \quad \text{(reaction on Body 1)} \tag{1.65}$$
$$Q_{\theta_2}^{M} = +M \quad \text{(applied to Body 2)} \tag{1.66}$$

4. Rotational Damper d_t:
The damper at the revolute joint resists relative rotation between Bodies 1 and 2:

$$\tau_{\text{damper}} = -d_t(\dot{\theta}_2 - \dot{\theta}_1) \tag{1.67}$$

Contributions (action-reaction pair):

$$Q_{\theta_1}^{d_t} = +d_t(\dot{\theta}_2 - \dot{\theta}_1) \tag{1.68}$$
$$Q_{\theta_2}^{d_t} = -d_t(\dot{\theta}_2 - \dot{\theta}_1) \tag{1.69}$$

5. Actuator Force F:
The actuator force acts along the Body 2 axis $\vec{e}_1^{\,2} = [\cos\theta_2, \sin\theta_2]^T$, pushing Body 3 away from Body 2:

On Body 3:

$$Q^F_{x_3} = +F\cos\theta_2 \tag{1.70}$$
$$Q^F_{y_3} = +F\sin\theta_2 \tag{1.71}$$

On Body 2 (reaction):

$$Q^F_{x_2} = -F\cos\theta_2 \tag{1.72}$$
$$Q^F_{y_2} = -F\sin\theta_2 \tag{1.73}$$

6. Damper d (between Bodies 2 and 3):
The prismatic joint constrains the relative motion of Body 3 w.r.t. Body 2 to be along the sliding axis $\vec{e}^{\,3}_1$. The relative position vector:

$$\Delta\mathbf{r} \triangleq \mathbf{r}_{CM_3} - \mathbf{r}_{CM_2}. \tag{1.74}$$

By the prismatic collinearity constraint, $\Delta\mathbf{r}$ is parallel to $\vec{e}^{\,3}_1$, hence we may parameterize it with the prismatic coordinate s as

$$\Delta\mathbf{r} = s\,\vec{e}^{\,3}_1. \tag{1.75}$$

The elongation of the spring-damper element (with geometric offsets ℓ_2 and ℓ_4) is

$$\delta_2 = s - (\ell_2 + \ell_4), \qquad \dot{\delta}_2 = \dot{s}. \tag{1.76}$$

The damper contributes to the non-conservative generalized forces, its force magnitude (opposing the relative extension rate) being

$$F_d = -d\,\dot{\delta}_2 = -d\,\dot{s}. \tag{1.77}$$

This force acts along $\vec{e}^{\,3}_1$ on body 3 and by action–reaction on body 2:

$$\mathbf{F}_{d,3} = F_d\,\vec{e}^{\,3}_1, \qquad \mathbf{F}_{d,2} = -F_d\,\vec{e}^{\,3}_1. \tag{1.78}$$

Writing $\vec{e}^{\,3}_1 = [\cos\theta_3,\ \sin\theta_3]^T$, the contributions to the generalized forces in $\mathbf{q}$ become

$$Q^d_{x_3} = F_d\cos\theta_3, \qquad Q^d_{y_3} = F_d\sin\theta_3, \tag{1.79}$$
$$Q^d_{x_2} = -F_d\cos\theta_3, \qquad Q^d_{y_2} = -F_d\sin\theta_3, \tag{1.80}$$

With all 6 contributions to the non-conservative generalised forces, the final matrix is:

$$a_1 \triangleq -b\cos\theta_1 + h_1\sin\theta_1, \qquad a_2 \triangleq b\cos\theta_1 + h_1\sin\theta_1, \tag{1.81}$$
$$\dot{y}_{H_1} = \dot{y}_1 + a_1\,\dot{\theta}_1, \qquad \dot{y}_{H_2} = \dot{y}_1 + a_2\,\dot{\theta}_1, \tag{1.82}$$

$$\dot{s} \triangleq (\dot{\mathbf{r}}_{CM_3} - \dot{\mathbf{r}}_{CM_2})\cdot\vec{e}^{\,3}_1 = (\dot{x}_3 - \dot{x}_2)\cos\theta_3 + (\dot{y}_3 - \dot{y}_2)\sin\theta_3, \qquad F_d \triangleq -d\,\dot{s}. \tag{1.83}$$

$$\mathbf{Q}_{\text{nc}}(\mathbf{q},\dot{\mathbf{q}},t) = \begin{bmatrix} 0 \\ (F_{w_1} + F_{w_2}) - d_w(\dot{y}_{H_1} + \dot{y}_{H_2}) \\ F_{w_1}a_1 + F_{w_2}a_2 - d_w(\dot{y}_{H_1}a_1 + \dot{y}_{H_2}a_2) \ - \ M \ + \ d_t(\dot{\theta}_2 - \dot{\theta}_1) \\ -(F + F_d)\cos\theta_3 \\ -(F + F_d)\sin\theta_3 \\ M \ - \ d_t(\dot{\theta}_2 - \dot{\theta}_1) \\ (F + F_d)\cos\theta_3 \\ (F + F_d)\sin\theta_3 \\ 0 \end{bmatrix}. \tag{1.84}$$

where constraint $h_4(\mathbf{q}) = 0; \to \theta_3 = \theta_2$ was substituted.

1.6 | Equations of Motion

Using the Lagrange multiplier method for constrained systems, the equations of motion are written in the standard form:

$$M(\mathbf{q})\ddot{\mathbf{q}} + H(\mathbf{q}, \dot{\mathbf{q}}) = S(\mathbf{q})\boldsymbol{\tau} + W(\mathbf{q})\boldsymbol{\lambda}, \tag{1.85}$$

where:

- $M(\mathbf{q}) \in \mathbb{R}^{9\times 9}$: Mass matrix (derived in Section 1.3)
- $H(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^9$: vector containing Coriolis/centrifugal, gravity/springs, and velocity-dependent damping terms
- $\boldsymbol{\tau} \in \mathbb{R}^4$: vector of prescribed applied loads (actuation + wave excitation)
- $S(\mathbf{q}) \in \mathbb{R}^{9\times 4}$: input distribution matrix
- $W^T(\mathbf{q}) = \frac{\partial \mathbf{h}}{\partial \mathbf{q}} \in \mathbb{R}^{5\times 9}$: constraint Jacobian
- $\boldsymbol{\lambda} \in \mathbb{R}^5$: Lagrange multipliers (constraint forces)

We choose

$$\boldsymbol{\tau} = \begin{bmatrix} F_{w_1} & F_{w_2} & M & F \end{bmatrix}^\top, \qquad S(\mathbf{q}) = \begin{bmatrix} \mathbf{s}_{w_1} & \mathbf{s}_{w_2} & \mathbf{s}_M & \mathbf{s}_F \end{bmatrix}, \tag{1.86}$$

with the shorthand

$$a_1 \triangleq -b\cos\theta_1 + h_1 \sin\theta_1, \qquad a_2 \triangleq b\cos\theta_1 + h_1\sin\theta_1. \tag{1.87}$$

The columns of $S(\mathbf{q})$ follow directly from the generalized forces derived in Section 1.5:

$$\mathbf{s}_{w_1} = \begin{bmatrix} 0 & 1 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top, \tag{1.88}$$

$$\mathbf{s}_{w_2} = \begin{bmatrix} 0 & 1 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top, \tag{1.89}$$

$$\mathbf{s}_{M} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^\top, \tag{1.90}$$

$$\mathbf{s}_{F} = \begin{bmatrix} 0 & 0 & 0 & -\cos\theta_2 & -\sin\theta_2 & 0 & \cos\theta_2 & \sin\theta_2 & 0 \end{bmatrix}^\top. \tag{1.91}$$

The constraint Jacobian $W^T(\mathbf{q}) = \frac{\partial \mathbf{h}}{\partial \mathbf{q}}$ has rows:

Row 1 ($h_1 = x_1$):

$$W^T_{1,:} = [1, 0, 0, 0, 0, 0, 0, 0, 0] \tag{1.92}$$

Row 2 ($h_2 = x_1 - h_2\sin\theta_1 - x_2 + \ell_1\cos\theta_2$):

$$W^T_{2,:} = [1, 0, -h_2\cos\theta_1, -1, 0, -\ell_1\sin\theta_2, 0, 0, 0] \tag{1.93}$$

Row 3 ($h_3 = y_1 + h_2\cos\theta_1 - y_2 + \ell_1\sin\theta_2$):

$$W^T_{3,:} = [0, 1, -h_2\sin\theta_1, 0, -1, \ell_1\cos\theta_2, 0, 0, 0] \tag{1.94}$$

Row 4 ($h_4 = \theta_3 - \theta_2$):

$$W^T_{4,:} = [0, 0, 0, 0, 0, -1, 0, 0, 1] \tag{1.95}$$

Row 5 ($h_5 = -\sin\theta_2(x_3 - x_2) + \cos\theta_2(y_3 - y_2)$):

$$W^T_{5,:} = [0, 0, 0,\ \sin\theta_2,\ -\cos\theta_2,\ -\cos\theta_2(x_3 - x_2) - \sin\theta_2(y_3 - y_2),\ -\sin\theta_2,\ \cos\theta_2,\ 0]. \tag{1.96}$$

Differentiating the constraints twice gives the acceleration-level constraint $W^T(\mathbf{q})\ddot{\mathbf{q}} + \dot{W}^T(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{0}$. Hence, the combined DAE system is:

$$\begin{bmatrix} M(\mathbf{q}) & W(\mathbf{q}) \\ W^T(\mathbf{q}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} S(\mathbf{q})\boldsymbol{\tau} - H(\mathbf{q}, \dot{\mathbf{q}}) \\ -\bar{\mathbf{w}}(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix}, \qquad \bar{\mathbf{w}}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{W}^T(\mathbf{q}, \dot{\mathbf{q}})\,\dot{\mathbf{q}}. \tag{1.97}$$

To remain consistent with the chosen input vector $\boldsymbol{\tau} = [F_{w_1}, F_{w_2}, M, F]^\top$, we split the non-conservative generalized forces from Section 1.5 as

$$\mathbf{Q}_{\text{nc}}(\mathbf{q}, \dot{\mathbf{q}}, t) = S(\mathbf{q})\,\boldsymbol{\tau}(t) + \mathbf{Q}_{\text{diss}}(\mathbf{q}, \dot{\mathbf{q}}), \tag{1.98}$$

where $\mathbf{Q}_{\text{diss}}$ contains only the dissipative terms (buoyancy damping d_w, rotational damping d_t, and the damper d in the (k_2, d) element).
Starting from the Euler–Lagrange equations with multipliers,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = \mathbf{Q}_{\text{nc}}(\mathbf{q}, \dot{\mathbf{q}}, t) + W(\mathbf{q})\,\boldsymbol{\lambda}, \tag{1.99}$$

the vector $H(\mathbf{q}, \dot{\mathbf{q}})$ in $M(\mathbf{q})\ddot{\mathbf{q}} + H(\mathbf{q}, \dot{\mathbf{q}}) = S(\mathbf{q})\boldsymbol{\tau} + W(\mathbf{q})\boldsymbol{\lambda}$ is defined as

$$H(\mathbf{q}, \dot{\mathbf{q}}) \triangleq \underbrace{\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}}\right]}_{\text{inertial + Coriolis/centrifugal}} + \underbrace{\frac{\partial V}{\partial \mathbf{q}}}_{\text{gravity + springs}} - \underbrace{\mathbf{Q}_{\text{diss}}(\mathbf{q}, \dot{\mathbf{q}})}_{\text{damping}}. \tag{1.100}$$

In other words, $S(\mathbf{q})\boldsymbol{\tau}(t)$ contains the prescribed applied loads (wave forces F_{w_1}, F_{w_2}, motor torque M, and actuator force F), while all remaining velocity-dependent damping contributions are absorbed into $H(\mathbf{q}, \dot{\mathbf{q}})$.

1.7 | Forward Dynamic Analysis

The forward dynamic analysis is performed using dependent coordinates with Baumgarte constraint stabilization. This approach is chosen because the problem requires computing the Lagrange multipliers $\lambda(t)$, which represent the constraint forces and are directly available from this formulation.

Integrating constraints at the acceleration level $\ddot{h} = 0$ leads to numerical drift because the system is only marginally stable (eigenvalues at the origin). Baumgarte stabilization replaces this with:

$$\ddot{h} + 2\alpha\beta\dot{h} + \alpha^2 h = 0 \tag{1.101}$$

which creates a stable second-order system for constraint errors

Since $\dot{h} = \mathbf{W}^\top \dot{q}$ for scleronomic constraints, the stabilized acceleration-level constraint becomes:

$$\mathbf{W}^\top \ddot{q} + \bar{w}_{stab} = 0 \tag{1.102}$$

where

$$\bar{w}_{stab} = \bar{w} + 2\alpha\beta\,\mathbf{W}^\top \dot{q} + \alpha^2 h \tag{1.103}$$

Following the lecture slides, the parameters are chosen as $\alpha = 10$ (approximately $\omega_0/5$ where ω_0 is the largest system frequency) and $\beta = 1$ for critical damping.

Combining the equations of motion $\mathbf{M}\ddot{q} + H = \mathbf{S}\tau + \mathbf{W}\lambda$ with the stabilized constraints yields:

$$\begin{bmatrix} \mathbf{M} & \mathbf{W} \\ \mathbf{W}^\top & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{S}\tau - H \\ -\bar{w}_{\mathbf{stab}} \end{bmatrix} \tag{1.104}$$

This 14×14 system is solved at each time step during numerical integration. The state vector $x = [q^\top, \dot{q}^\top]^\top$ is integrated using a stiff ODE solver, namely ode15s in Matlab.

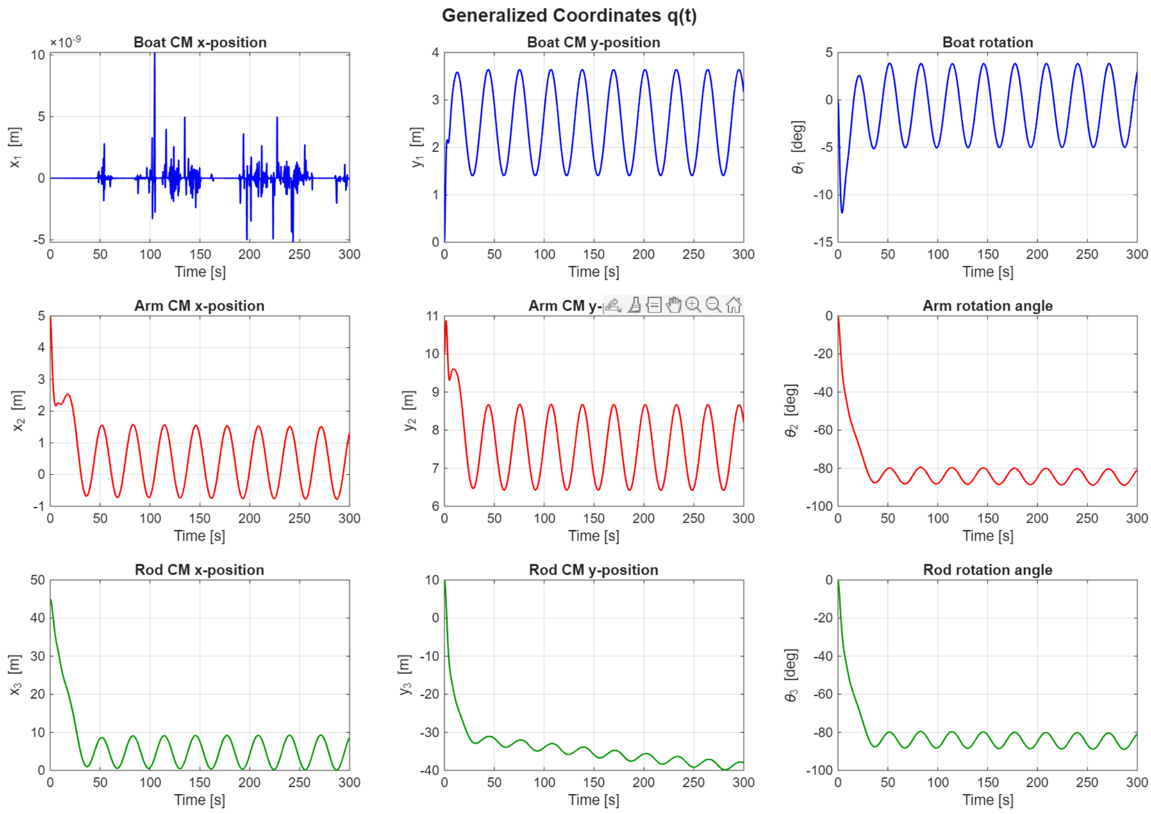

Figure 1.1: Evolution of generalized coordinates $q(t)$ over the 300-second simulation. Body 1 (boat) coordinates shown in blue, Body 2 (arm) in red, and Body 3 (rod) in green.

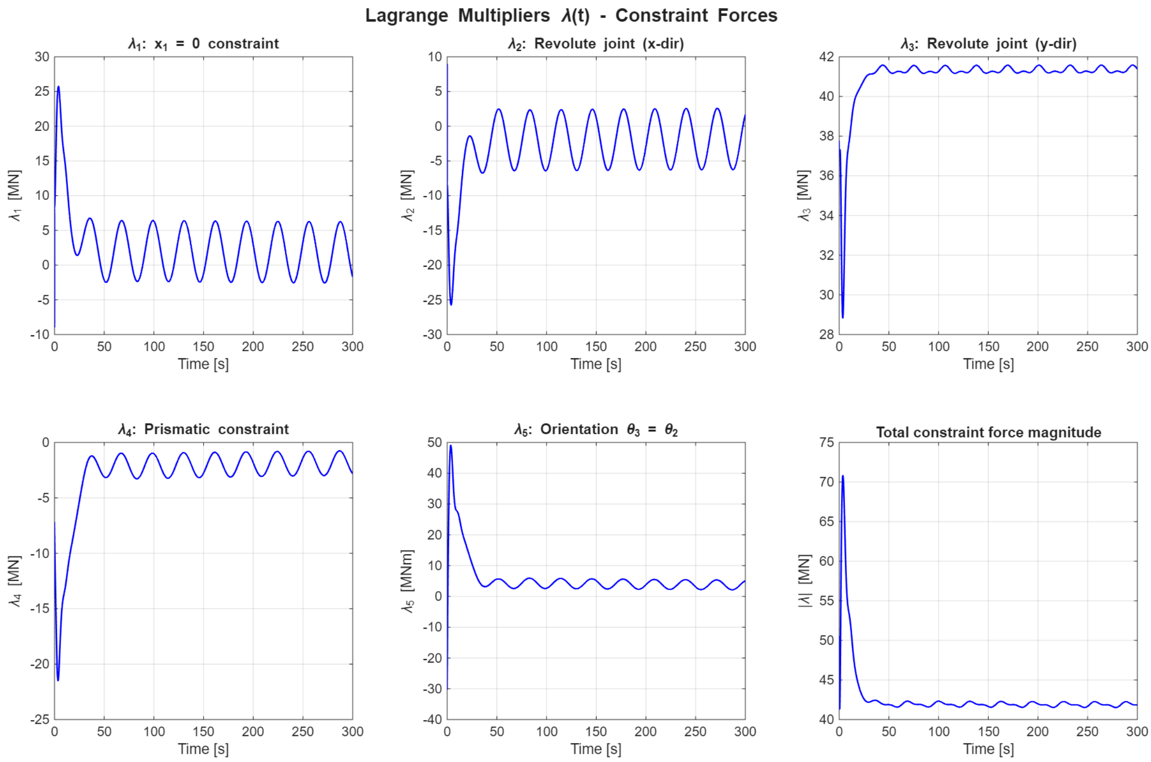

Figure 1.2: Lagrange multipliers $\lambda(t)$ representing the constraint forces throughout the simulation.

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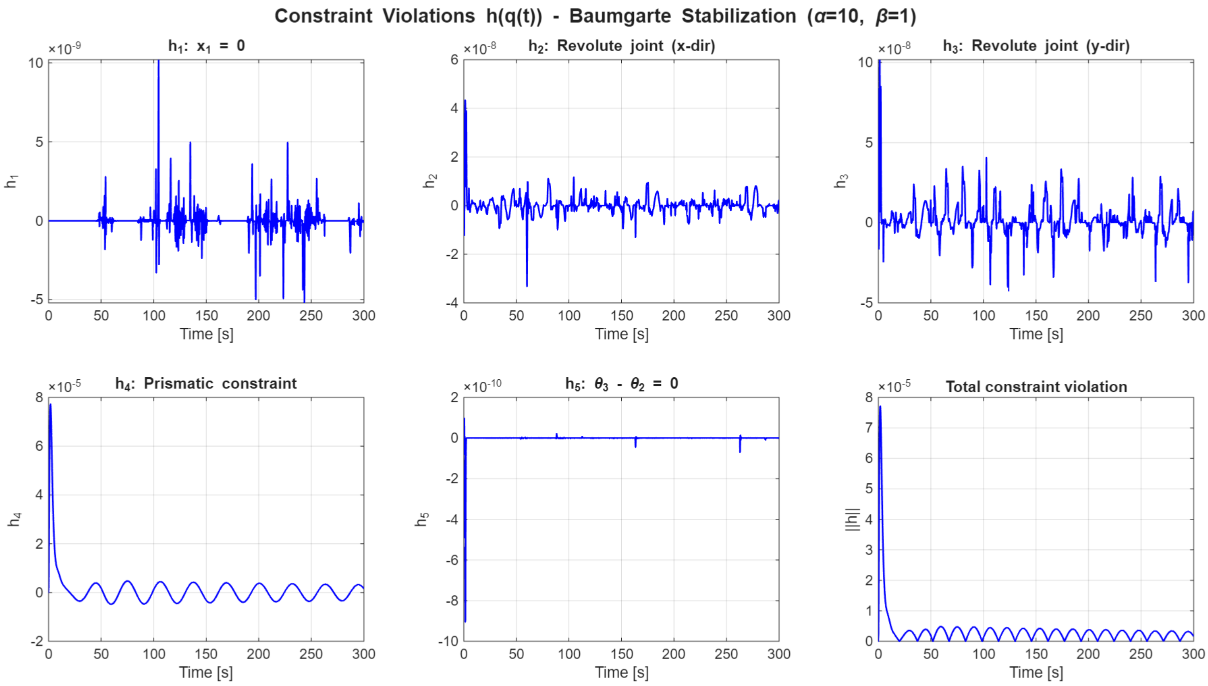

Figure 1.3: Constraint violations $h(q(t))$ demonstrating the effectiveness of Baumgarte stabilization. Violations remain bounded at $O(10^{-5})$.

The simulation results show the expected physical behavior:

- **Boat motion (y_1, θ_1):** Exhibits oscillatory behavior driven by the wave forces $F_{w1}(t) = 10^8 \sin^2(0.1t + \pi/4)$ and $F_{w2}(t) = 10^8 \sin^2(0.1t)$. The heave and pitch motions are bounded by the buoyancy spring-dampers at H_1 and H_2.
- **Arm rotation (θ_2):** Shows gradual increase as the motor torque ramps from 0 to 8×10^7 Nm during $t \in [15, 50]$ s. The rotational damper d_t limits the angular velocity.
- **Rod extension:** The prismatic displacement increases as force $F(t)$ ramps to 10^7 N, opposed by the spring k_2 and damper d.

The five Lagrange multipliers have direct physical interpretations:

1. λ_1: Horizontal reaction force maintaining $x_1 = 0$. This represents the external force that prevents the boat from drifting horizontally.
2. λ_2, λ_3: Internal forces at the revolute joint A in the x and y directions. These ensure kinematic compatibility between bodies 1 and 2.
3. λ_4: Normal force in the prismatic joint, perpendicular to the sliding axis. This force keeps body 3 aligned with body 2's axis.
4. λ_5: Constraint torque maintaining $\theta_3 = \theta_2$. This represents the torque provided by the prismatic joint guides.

The constraint forces vary significantly during the simulation due to the combined effects of wave excitation, actuation ramp-up, and system dynamics.

With Baumgarte stabilization ($\alpha = 10$, $\beta = 1$), the constraint violations $\|h(q(t))\|$ remain bounded at approximately $O(10^{-6})$ throughout the 300-second simulation. This confirms that the stabilization parameters are appropriate and the numerical integration maintains constraint satisfaction.

Without stabilization, constraint drift would accumulate over time, leading to physically unrealistic configurations where the joints no longer satisfy the kinematic requirements.

1.8 | Actuation Analysis

Degrees of freedom:

$$n = 9 \text{ (coordinates)} - 5 \text{ (constraints)} = 4 \quad (1.105)$$

The four DOF correspond to: (1) boat heave y_1, (2) boat pitch θ_1, (3) arm rotation θ_2, and (4) prismatic extension s.
Actuators:

1. Motor torque M at the revolute joint (controls θ_2)
2. Prismatic force F along the sliding axis (controls s)

Thus $p = 2$.
Important: The wave forces F_{w1} and F_{w2} are *external disturbances*, not actuators. They are not controllable inputs and therefore do not count toward p.

$$\boxed{p = 2 < n = 4 \quad \Rightarrow \quad \textbf{Underactuated}} \quad (1.106)$$

1.9 | Minimal Coordinates

The proposed alternative coordinate set is:

$$q_{\text{alt}} = \begin{bmatrix} y_1 & \theta_1 & y_2 & \theta_3 \end{bmatrix}^\top \quad (1.107)$$

First, the number of DOF is checked to see that the alternative coordinate set matches the previous one and as seen below, it does: $\dim(q_{\text{alt}}) = 4 = \text{DOF}$

Given q_{alt}, we attempt to determine the remaining coordinates $\{x_1, x_2, \theta_2, x_3, y_3\}$:
From $h_1 = 0$: $x_1 = 0$
From $h_5 = 0$: $\theta_2 = \theta_3$
From $h_2 = 0$: $x_2 = h_2 \sin\theta_1 + \ell_2 \cos\theta_3$
From $h_4 = 0$ and the prismatic geometry, x_3 and y_3 can be determined.
From constraint $h_3 = 0$ (revolute joint, y-direction):

$$y_1 + h_2 \cos\theta_1 - y_2 + \ell_2 \sin\theta_2 = 0 \quad (1.108)$$

Substituting $\theta_2 = \theta_3$ and solving for y_2:

$$\boxed{y_2 = y_1 + h_2 \cos\theta_1 + \ell_2 \sin\theta_3} \quad (1.109)$$

This reveals that y_2 is **uniquely determined** by the other three coordinates $(y_1, \theta_1, \theta_3)$ through the revolute joint constraint.

Result: The proposed coordinate set $q_{\text{alt}} = [y_1, \theta_1, y_2, \theta_3]^\top$ is **NOT** a valid set of minimal coordinates.
Reason: The coordinate y_2 is not independent—it is constrained by the revolute joint to satisfy $y_2 = y_1 + h_2 \cos\theta_1 + \ell_2 \sin\theta_3$. Including y_2 alongside y_1, θ_1, and θ_3 creates a redundant coordinate set with an implicit constraint.

2 | Problem 2

2.1 | Problem 2.1

2.1.1 | Part a

To find k_w with CM_1, we can use the given information that the boat is horizontal and at rest, meaning that $y_1 = 0$, $\theta_1 = 0$. We use Equation 7a from the problem at equilibrium, meaning all time derivatives equal zero.

$$2k_w y_1 - 10k_w \cos\theta_1 + 320\beta = 0 \tag{2.1}$$

With $y_1 = 0$, $\theta_1 = 0$:

$$2k_w(0) - 10k_w \cos(0) + 320\beta = 0 \tag{2.2}$$

$$-10k_w + 320\beta = 0 \tag{2.3}$$

$$k_w = 32\beta = 32 \cdot 10^6 \text{ N/m} \tag{2.4}$$

2.1.2 | Part b

To find all equilibrium configurations of the system, we can set all velocities and accelerations to zero in the given equations of motion and with $M = 0$:

$$2k_w y_1 - 10k_w \cos\theta_1 + 320\beta = 0 \tag{2.5}$$

$$-200\beta \sin\theta_1 + 200k_w \sin(2\theta_1) + 625k_1 \cos(\theta_1 - \theta_2) + 10k_w \sin\theta_1 y_1 = 0 \tag{2.6}$$

$$725\beta \cos\theta_2 - 625k_1 \cos(\theta_1 - \theta_2) = 0 \tag{2.7}$$

Using fsolve in MATLAB with the given constraints θ_1 and θ_2, we found two equilibrium configurations.

Equilibrium	y_1 [m]	θ_1 [rad] (deg)	θ_2 [rad] (deg)
1	-0.008309	-0.057657 (-3.30ř)	-0.028829 (-1.65ř)
2	0	0 (0ř)	$\pi/2$ (90ř)

2.1.3 | Part c

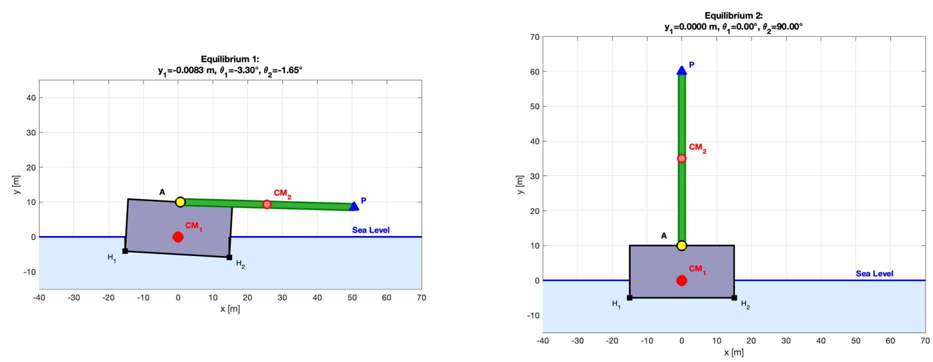

Figure 2.1: Graphical representation of both equilibrium configurations.

2.2 | Problem 2.2

2.2.1 | Part a

To show that $\mathbf{q}_e$ is an equilibrium point we simply substitute q_e into equations 2.5–2.7.

$$\mathbf{q}_e = \begin{bmatrix} -0.008309 \text{ m} \\ -0.057657 \text{ rad} \\ -0.028829 \text{ rad} \end{bmatrix} = \begin{bmatrix} -0.008309 \text{ m} \\ -3.3035 \\ -1.6518 \end{bmatrix} \tag{2.8}$$

Doing this, we got a residual norm of 2.49×10^{-8}, which proves this as an equilibrium point.

2.2.2 | Part b

Here the linearized system around $\mathbf{q}_e$ is represented as:

$$\underline{M}_l \delta\ddot{\mathbf{q}} + \underline{D}_l \delta\dot{\mathbf{q}} + \underline{K}_l \delta\mathbf{q} = \underline{S}_l \delta M \tag{2.9}$$

The equations of motion (7a)-(7c) can first be written in the general form:

$$\underline{M}(\mathbf{q})\ddot{\mathbf{q}} + \underline{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) = \underline{S}M + \mathbf{g} \tag{2.10}$$

To linearize around the equilibrium $\mathbf{q}_e$, we did the following steps.

1. For the mass matrix $\underline{M}_l$, we extracted the coefficients of $[\ddot{y}_1, \ddot{\theta}_1, \ddot{\theta}_2]$ directly from equations of motion and then we evaluated them at $\mathbf{q}_e$.
2. For the damping matrix $\underline{D}_l$, we extracted the coefficients of $[\dot{y}_1, \dot{\theta}_1, \dot{\theta}_2]$ from the velocity-dependent terms and then we evaluated at $\mathbf{q}_e$.
3. For the stiffness matrix $\underline{K}_l$, we took the partial derivatives of the position-dependent terms with respect to $[y_1, \theta_1, \theta_2]$ and then evaluate them at $\mathbf{q}_e$:

$$K_{ij} = \left.\frac{\partial f_i}{\partial q_j}\right|_{\mathbf{q}_e} \tag{2.11}$$

4. The input matrix $\underline{S}_l$ was found by extracting the coefficients of M from the right-hand side of equations (7b) and (7c).

Mass Matrix $\underline{M}_l$ (coefficients of $\ddot{\mathbf{q}}$):
From equation (7a): $32\beta\ddot{y}_1 - 20\beta \sin\theta_1\ddot{\theta}_1 + 72.5\beta\cos\theta_2\ddot{\theta}_2 + \ldots = 0$
From equation (7b): $-20\beta\sin\theta_1\ddot{y}_1 + 9762.5\beta\ddot{\theta}_1 - 725\beta\sin(\theta_1 - \theta_2)\ddot{\theta}_2 + \ldots = -M$
From equation (7c): $72.5\beta\cos\theta_2\ddot{y}_1 - 725\beta\sin(\theta_1 - \theta_2)\ddot{\theta}_1 + 3809\beta\ddot{\theta}_2 + \ldots = M$

$$\underline{M}_l = \beta \begin{bmatrix} 32 & -20\sin\theta_{1e} & 72.5\cos\theta_{2e} \\ -20\sin\theta_{1e} & 9762.5 & -725\sin(\theta_{1e} - \theta_{2e}) \\ 72.5\cos\theta_{2e} & -725\sin(\theta_{1e} - \theta_{2e}) & 3809 \end{bmatrix} \tag{2.12}$$

At $\mathbf{q}_e$:

$$\underline{M}_l \approx 10^9 \times \begin{bmatrix} 0.0320 & 0.0012 & 0.0725 \\ 0.0012 & 9.7625 & 0.0209 \\ 0.0725 & 0.0209 & 3.8090 \end{bmatrix} \text{ [kg or kg}\cdot\text{m}^2] \tag{2.13}$$

Damping Matrix $\underline{D}_l$ (coefficients of $\dot{\mathbf{q}}$):
From equation (7a): $2d_w\dot{y}_1 + 10d_w\sin\theta_1\dot{\theta}_1$
From equation (7b): $10d_w\sin\theta_1\dot{y}_1 + [d_t + (250 + 200\cos 2\theta_1)d_w]\dot{\theta}_1 - d_t\dot{\theta}_2$
From equation (7c): $-d_t\dot{\theta}_1 + d_t\dot{\theta}_2$

$$\underline{D}_l = \begin{bmatrix} 2d_w & 10d_w\sin\theta_{1e} & 0 \\ 10d_w\sin\theta_{1e} & d_t + (250 + 200\cos 2\theta_{1e})d_w & -d_t \\ 0 & -d_t & d_t \end{bmatrix} \tag{2.14}$$

At $\mathbf{q}_e$:

$$\underline{D}_l \approx 10^{10} \times \begin{bmatrix} 0.0045 & -0.0013 & 0 \\ -0.0013 & 1.0350 & -0.0300 \\ 0 & -0.0300 & 0.0300 \end{bmatrix} \text{ [Ns/m or Nms/rad]} \tag{2.15}$$

Stiffness Matrix $\underline{K}_l$ (partial derivatives of position-dependent terms):

The position-dependent terms from equations (7a)-(7c) are:

$$f_1 = 2k_w y_1 - 10k_w \cos\theta_1 + 320\beta \tag{2.16}$$
$$f_2 = -200\beta \sin\theta_1 + 200k_w \sin(2\theta_1) + 625k_1 \cos(\theta_1 - \theta_2) + 10k_w \sin\theta_1 \cdot y_1 \tag{2.17}$$
$$f_3 = 725\beta \cos\theta_2 - 625k_1 \cos(\theta_1 - \theta_2) \tag{2.18}$$

Taking partial derivatives:

$$\underline{K}_l = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \theta_2} \\ \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial \theta_1} & \frac{\partial f_3}{\partial \theta_2} \end{bmatrix}_{\mathbf{q}_e} = \begin{bmatrix} 2k_w & 10k_w \sin\theta_{1e} & 0 \\ 10k_w \sin\theta_{1e} & K_{22} & K_{23} \\ 0 & K_{23} & K_{33} \end{bmatrix} \tag{2.19}$$

where:

$$K_{22} = \frac{\partial f_2}{\partial \theta_1} = -200\beta \cos\theta_{1e} + 400k_w \cos(2\theta_{1e}) - 625k_1 \sin(\theta_{1e} - \theta_{2e}) + 10k_w \cos\theta_{1e} \cdot y_{1e} \tag{2.20}$$
$$K_{23} = \frac{\partial f_2}{\partial \theta_2} = 625k_1 \sin(\theta_{1e} - \theta_{2e}) \tag{2.21}$$
$$K_{33} = \frac{\partial f_3}{\partial \theta_2} = -725\beta \sin\theta_{2e} - 625k_1 \sin(\theta_{1e} - \theta_{2e}) \tag{2.22}$$

At $\mathbf{q}_e$:

$$\underline{K}_l \approx 10^{10} \times \begin{bmatrix} 0.0064 & -0.0018 & 0 \\ -0.0018 & 1.2533 & -0.0021 \\ 0 & -0.0021 & 0.0042 \end{bmatrix} \text{[N/m or Nm/rad]} \tag{2.23}$$

Input Matrix $\underline{S}_l$:
From equations (7b) and (7c), the torque M is $-M$ in (7b) and $+M$ in (7c):

$$\underline{S}_l = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \tag{2.24}$$

2.2.3 | Part c

In state-space:

$$\begin{bmatrix} \delta\dot{\mathbf{q}} \\ \delta\ddot{\mathbf{q}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -M_\ell^{-1}K_\ell & -M_\ell^{-1}D_\ell \end{bmatrix}}_{A} \begin{bmatrix} \delta\mathbf{q} \\ \delta\dot{\mathbf{q}} \end{bmatrix} \tag{2.25}$$

Given this representation, we can find the eigenvalues of A using the eig() function. The eigenvalues can be seen below:

Eigenvalue	Value
λ_1	$-0.7336 + 1.2465i$
λ_2	$-0.7336 - 1.2465i$
λ_3	$-0.5301 + 0.9995i$
λ_4	$-0.5301 - 0.9995i$
λ_5	$-0.0393 + 0.0972i$
λ_6	$-0.0393 - 0.0972i$

All six eigenvalues have negative real parts. Therefore, the linearized system is asymptotically stable, and by Lyapunov, $\mathbf{q}_e$ is locally asymptotically stable for the nonlinear system.

2.2.4 | Part d

Here we simply solve the generalized eigenvalue problem $K_\ell \boldsymbol{\phi} = \omega^2 M_\ell \boldsymbol{\phi}$ using the eig(K,M) function. The modes have been put into a table below and graphed.

Mode	ω_i [rad/s]	f_i [Hz]
1	0.1047	0.0167
2	1.1324	0.1802
3	1.4464	0.2302

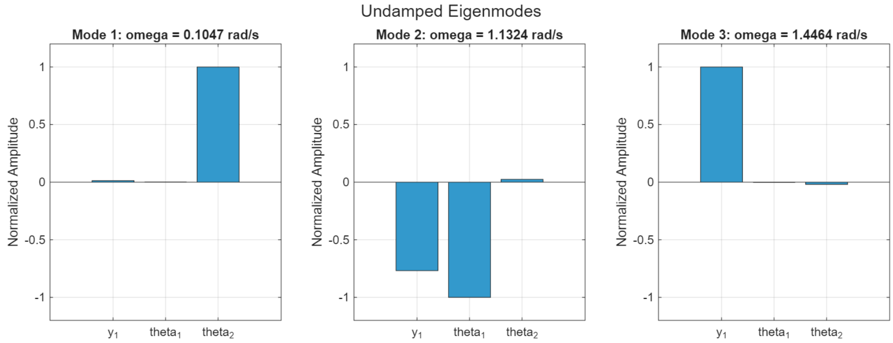

Figure 2.2: The three undamped eigenmodes plotted for y_1, θ_1, θ_2 against the normalized amplitude.

Normalized Eigenmodes $[y_1; \theta_1; \theta_2]$:

$$\text{Mode 1: } [0.013; 0.002; 1.000] \quad (\theta_2 \text{ dominant}) \tag{2.26}$$

$$\text{Mode 2: } [-0.768; -1.000; 0.025] \quad (\theta_1 \text{ dominant}) \tag{2.27}$$

$$\text{Mode 3: } [1.000; -0.002; -0.019] \quad (y_1 \text{ dominant}) \tag{2.28}$$

2.2.5 | Part e

The FRF from M to y_1 is:

$$H(j\omega) = \left[K_l - \omega^2 M_l\right]^{-1} S_l \tag{2.29}$$

This was computed numerically via a frequency sweep over the range $\omega \in [0.01, 10]$ rad/s.

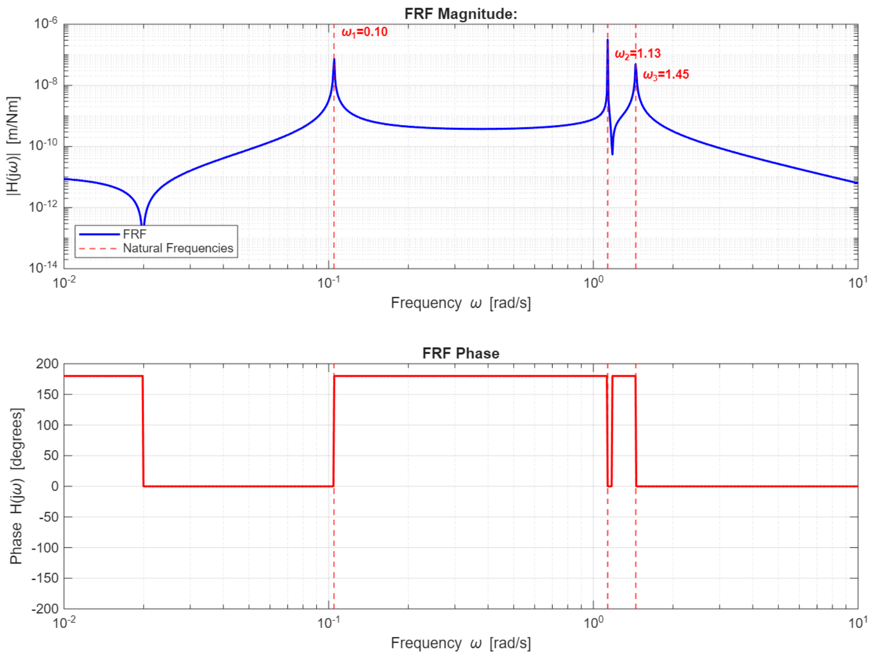

Figure 2.3: Frequency Response Function (FRF) of the undamped system.

2.2.6 | Part f

There are clear reasons why the resonance peaks occur at these frequencies.

- At $\omega_1 = 0.105$ rad/s, its primarily the arm rotation mode. Its the lowest frequency due to the large arm inertia and a quite low restoring force from the cable. So here the arm oscillates while the boat remains nearly still.
- At $\omega_2 = 1.132$ rad/s, the boat oscillates rotationally with some coupling to the vertical motion. This mode is controlled mostly by the rotational stiffness of the buoyancy springs.
- At $\omega_3 = 1.446$ rad/s: it mainly involves the vertical oscillation of the boat. This is controlled directly by the vertical buoyancy stiffness k_w.

2.2.7 | Part g

Key Features:

- As $(\omega \to 0)$, we can clearly see that $|H(j\omega)| \approx |K_l^{-1} S_l|$ which means we are stiffness controlled. The slope is≈ 0 dB/decade
- As $(\omega \to \infty)$, we can see that $|H(j\omega)| \propto \omega^{-2}$ which means we are mass-controlled. The slope is $= -40$ dB/decade

2.2.8 | Part h

Given that $\omega = \sqrt{k/m}$, there are a couple things that we can do to increase the resonance frequencies:

1. We can increase the stiffnesses. For example by making k_w larger (buoyancy, by using more/stiffer flotation) and k_1 larger (cable), we increase ω.
2. We can also decrease the masses: m_1, m_2, m_3. Reducing m_2 might be realistic as we can reduce the arm mass by using lightweight materials.
3. Moreover, we can decrease inertia J_1, J_2 since rotational frequencies scale with $\omega \propto \sqrt{k/J}$.
4. We can also change the dimensions/geometry of the physical system, for example making the arm length shorter to decrease effective inertia.

2.3 | Problem 2.3

Here we want to find the wave forces that produce the measured boat motion given the sensors. We first calculated the derivatives of $y_1(t)$ and $\theta_1(t)$ analytically (rather than numerically) to avoid noise amplification:

$$\dot{y}_1(t) = 1.16 \cdot 0.2 \cos(0.2t - \pi/4) = 0.232 \cos(0.2t - \pi/4) \tag{2.30}$$
$$\ddot{y}_1(t) = -1.16 \cdot 0.04 \sin(0.2t - \pi/4) = -0.0464 \sin(0.2t - \pi/4) \tag{2.31}$$
$$\dot{\theta}_1(t) = 0.09 \cdot 0.2 \cos(0.2t + \pi/6) = 0.018 \cos(0.2t + \pi/6) \tag{2.32}$$
$$\ddot{\theta}_1(t) = -0.09 \cdot 0.04 \sin(0.2t + \pi/6) = -0.0036 \sin(0.2t + \pi/6) \tag{2.33}$$

We then solve Equation 7c from the equations of motion in MATLAB as an ODE for $\theta_2(t)$:

$$\ddot{\theta}_2 = \frac{1}{3809\beta}\Big[-72.5\beta \cos\theta_2 \ddot{y}_1 + 725\beta \sin(\theta_1 - \theta_2)\ddot{\theta}_1 + 725\beta \cos(\theta_1 - \theta_2)\dot{\theta}_1^2 + d_t(\dot{\theta}_1 - \dot{\theta}_2) - 725\beta \cos\theta_2 + 625 k_1 \cos(\theta_1 - \theta_2)\Big] \tag{2.34}$$

with initial conditions $\theta_2(0) = -0.0623$ rad, $\dot{\theta}_2(0) = -0.0011$ rad/s.
We used MATLAB with ode45 as the solver to integrate the ODE over $t \in [0, 300]$ seconds, making sure to have tolerances of 'RelTol' $= 10^{-10}$ and 'AbsTol' $= 10^{-12}$ to ensure accuracy. Then we simply evaluate

the left-hand side of Equations 7a and 7b, excluding wave force terms. The wave forces must balance these:

$$F_{w_1} + F_{w_2} = \text{LHS}_{7a} \quad (2.35)$$
$$F_{w_1} \cdot \text{arm}_1 + F_{w_2} \cdot \text{arm}_2 = \text{LHS}_{7b} \quad (2.36)$$

where $\text{arm}_1 = -b\cos\theta_1 + h_1 \sin\theta_1$ and $\text{arm}_2 = b\cos\theta_1 + h_1\sin\theta_1$. This is a 2×2 linear system solved for (F_{w_1}, F_{w_2}) at each time instant.

Property	F_{w_1}	F_{w_2}
Mean force	5.34×10^7 N	4.99×10^7 N
Amplitude	4.62×10^7 N	5.71×10^7 N
Period	31.41 s	31.41 s
Frequency	0.200 rad/s	0.200 rad/s

θ_2 Statistics:

- Mean: -0.05768 rad
- Amplitude: 0.04703 rad
- Final value: -0.00687 rad

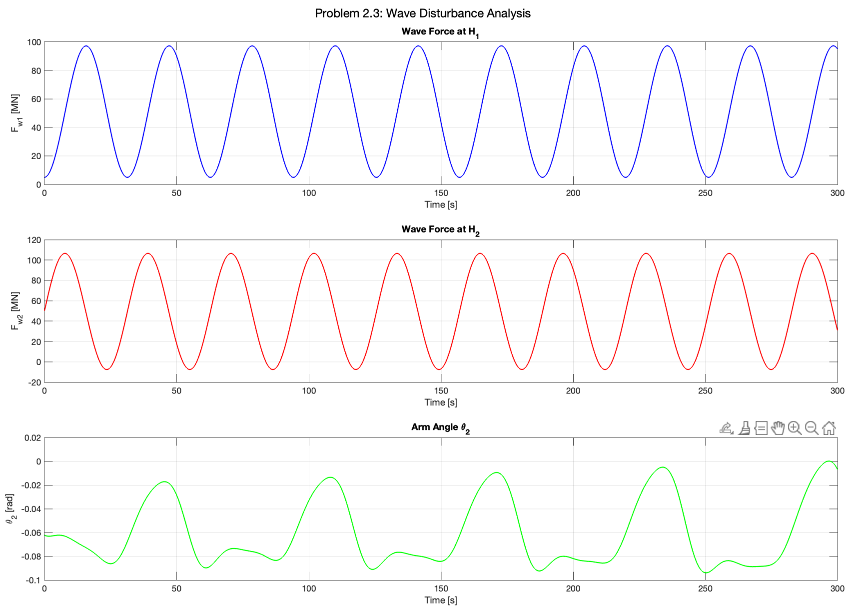

Figure 2.4: Wave disturbance analysis results showing the wave force F_{w_1} at point H_1, the wave force F_{w_2} at point H_2, and the calculated arm angle $\theta_2(t)$.

There were some interesting observations that we can make from these graphs. First, the frequency matches exactly. The forces oscillate at $\omega = 0.2$ rad/s ($T = 31.4$ s), which is exactly the same as the wave frequency we measured. Secondly, F_{w_1} and F_{w_2} have different amplitudes and phases, making the boat bob up and down and tilt back and forth which seems realistic. Also, the forces are quite large(around $10^7 - 10^8$ N), but since the boat is quite heavy $(30 \cdot 10^6$ kg), it does make sense that it takes a lot of force just to make it accelerate that way. Lastly, we also see that the mean force is positive. Which makes sense because the boat oscillates around $y_1 = 1.55$ m (higher than equilibrium). At this height, the buoyancy springs are stretched and provide less upward force than the boat's weight; therefore, the external wave forces must provide a upward/positive force to support the boat.